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On the Absence of Breakdown of Symmetry for the Plane Rotator Model with Long-Range Random Interaction

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We study the plane rotator model with hamiltonian

$$-\frac{1}{2}\sum_{x\neq y} J_{xy}\frac{\cos(\theta_x-\theta_y)}{|x-y|^{3+\epsilon}}$$

where J_{xy} for different pair (x, y) are independent symmetric random variables. It is proved that for almost all J, *all* the Gibbs states $P(J)$ are rotation invariant.

KEY WORDS: Random interaction; random variable long range; spin glass; relative entropy.

1. INTRODUCTION

A spin glass is a dilute magnetic alloy where magnetic impurities, say Fe, are dilute in a nonmagnetic metal, say Au. Experimentally the susceptibility have a cusp at some temperature T_{SG}.[1] It is believed[2] that this comes from the Ruderman–Kittel–Kasuya–Yosida (RKKY) spin–spin interaction of the impurities. This is given by the formula

$$H(x, y) = J(x, y)\mathbf{S}(x)\cdot\mathbf{S}(y) \tag{1.1}$$

where

$$J(x, y) = \frac{A}{|x-y|^3}\cos(2k_F|x-y|)$$

$\mathbf{S}(x)$ is the spin of the impurity and k_F the Fermi momentum. The RKKY

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interaction is long range and rapidly oscillating. The first simplification of this model comes from Edwards and Anderson,[3] who argue that this is the *oscillating* of the exchange interaction which is essential to produce a spin glass. They introduce a model with *random* short-range interaction: the $J(x, y)$ were Gaussian random variables. They introduce two order parameters which characterize a spin glass phase: Let $\langle\ \rangle(J)$ denote expectation with respect to a Gibbs state corresponding to a *given* configuration of exchange interactions J. Let $\mathbb{E}$ denote the expectation with respect to the random variables J. The first parameter is the mean magnetization $\mathbb{E}[\langle S_3\rangle(J)]$ which is assumed to be zero. The second parameter is $q_{\mathrm{EA}} = \mathbb{E}[\langle \mathbf{S}\rangle(J) \cdot \langle \mathbf{S}\rangle(J)]$, which is strictly positive in a spin glass phase.

Historically the first investigation was about mean field theory: Edwards and Anderson (EA)[3] predict a spin glass phase. Sherrington and Kirkpatrick (SK)[4] define a random Curie–Weiss theory which predicts also a spin glass phase but leads to negative entropy at low temperature. The use of the n-replica trick has been suggested as cause of this unphysical phenomenon. An alternative approach to this problem was proposed by Thouless, Anderson, and Palmer (TAP)[5] in which the n-replica trick is avoided. The so-called TAP equation for the two previous order parameters of EA shows a spin glass phase for the SK model. There exists a rigorous proof of this fact in Thompson.[6]

For a more realistic model of spin glass, the existence of a phase transition from conventional phases into a spin glass phase is much less clear. One important question is the lower critical dimension D_0 for spin glass phase i.e., if $D > D_0$ there is a spin glass phase and if $D \leqslant D_0$ there is no spin glass phase (in the case of nearest-neighbor interactions). There are many controversies on the subject (see the paper of S. Kirkpatrick in Ref. 7 and Villain in Ref. 7). Most of the authors expect that $D_0 = 2$ or 3 of the Ising model and $D_0 = 3$ or 4 for a model with continuous internal symmetry such as the classical x–y model (see Refs. 8–13).

Rigorous results on spin glass are few: Vuillermot[14] proved that the infinite-volume limit of the free energy (with free boundary conditions) is almost surely equal to the infinite-volume limit of the quenched free energy [i.e., $(1/|\Lambda|)\mathbb{E}(\log Z_\Lambda)$]. Vuillermot gave abstract conditions on random variables J. In the case $J(x, y) = J_{xy}/|x-y|^{\alpha d}$ where d is the dimension of the lattice and J_{xy} are independent random variables, say Bernoulli symmetric random variables, his condition is equivalent to $\alpha > 1$. This is the usual condition to obtain a well-defined infinite-volume free energy.

Ledrappier in Ref. 15 proved similar results in the case of the nearest-neighbor Ising model. He proved also a variational principle in an abstract case where J need not be independent.

Khanin and Sinaï[16] proved the *stronger* result that if $\alpha > 1/2$ then the infinite-volume free energy exists almost surely, is independent of the boundary condition, and is almost surely equal to the infinite volume quenched free energy. It is asserted (without proof) that if $\alpha \leqslant 1/2$ infinite-volume free energy is almost surely divergent. The proof is given for the Ising system, and it is asserted, without proof, that the same result is true for the bounded continuous spin model.

There are also results related to the uniqueness of Gibbs states:

In one dimension and *long-ranged* Ising model Khanin[17] proved that if $\alpha = 3/2 + \epsilon$ then for almost all J there is only one Gibbs state $P(J)$. For the same model Cassandro, Olivieri and Tirozzi[18] proved that the infinite-volume free energy is almost surely C^∞ in β and in the magnetic field h. They proved the same result for the infinite-volume quenched free energy. Remark that in the case where all $J_{xy} = 1$ the previous results are false: there exists spontaneous magnetization.[19] The random character of the interaction is crucial.

In the two-dimensional nearest-neighbors Ising spin glass Avron, Roepstorff, and Schulman[20] proved that the first parameter of Edwards and Anderson vanished; in fact they proved more: for any

$$A \subset \mathbb{Z}^2 \qquad \mathbb{E}\left[\left\langle \prod_{x \in A} \sigma_x \right\rangle (J)\right] = 0$$

In one and two dimensions Vuillermot[13] proved, by using a Bogoliubov-type inequality, that in models with continuous internal symmetry there is no *mean* ordering. He did not prove that the second-order parameter of EA (q_{EA}) vanishes but rather that almost surely

$$\lim_{\Lambda \to \mathbb{Z}^2} \left[\frac{1}{|\Lambda|} \sum_{x \in \Lambda} \langle S_x^{(3)} \rangle (J) \right]^2 = 0 \tag{1.2}$$

$$\lim_{\Lambda \to \mathbb{Z}^2} \mathbb{E}\left[\frac{1}{|\Lambda|} \sum_{x \in \Lambda} \langle S_x^{(3)} \rangle (J) \right]^2 = 0 \tag{1.3}$$

where $S_x^{(3)}$ is the 3-component of the spin $\mathbf{S}_x$. A more realistic parameter for spin glass phases is, as quoted by Vuillermot,

$$\lim_{\Lambda \to \mathbb{Z}^2} \frac{1}{|\Lambda|} \left[\sum_{x \in \Lambda} \langle S_x^{(3)} \rangle^2 (J) \right] \quad \text{or} \quad \lim_{\Lambda \to \mathbb{Z}^2} \frac{1}{|\Lambda|} \mathbb{E}\left[\sum_{x \in \Lambda} \langle S_x^{(3)} \rangle^2 (J) \right] \tag{1.4}$$

On the other hand, in the two-dimensional case with long-range exchange $J(x, y) = J_{xy}/|x-y|^\alpha$ with J_{xy} independent symmetric Bernoulli random variables, Vuillermot condition for zero mean ordering is $\alpha \geqslant 4 + \epsilon$. Let us remark that if $\alpha \geqslant 4$, Pfister and Fröhlich[21,22] have proved that there is no breakdown of the symmetry and the sign of J_{xy} can be arbitrary. There theorem applies in the spin glass case, if the J_{xy} are bounded random variables.

In this paper we study the classical (x–y) spin glass model in two dimensions with long-range exchange interaction $J(x, y) = J_{xy}/|x-y|^{3+\epsilon}$. By using a random version of the relative entropy argument of Fröhlich and Pfister,[22] we prove that for *almost all J* there is no breakdown of the symmetry. This is a stronger result than the vanishing of the EA parameter q_{EA}. Remark that if all J_{xy} are equal to unity there is breakdown of the symmetry (see Kunz and Pfister[23]).

Since, as quoted by most authors (Anderson[7] or Kirkpatrick[7]), the energy $H_\Lambda(J) \approx \{ [H_\Lambda^2(J)]\}^{1/2}$ the reader can wonder why the argument does not apply when $J(x, y) = J_{xy}/|x-y|^{2+\epsilon}$ because in this case the Hamiltonian is "equivalent" to the ferromagnetic one with exchange interaction $J(x, y) = 1/|x-y|^{4+2\epsilon}$, which is the usual condition for absence of breakdown of symmetry. The same kind of argument implies that in the three-dimensional nearest-neighbor classical x–y spin glass model there is no breakdown of the symmetry. In fact $H_\Lambda(J, \sigma_\Lambda, \sigma_{\Lambda^c}) \approx \mathbb{E}[h_\Lambda^2(J, \sigma_\Lambda, \sigma_{\Lambda^c})]^{1/2}$ is true (as the law of iterated logarithm for tail sums[24] asserts) for a *given* configuration of the spins.

We have to consider *simultaneously all* the (*strongly* dependent) random variables $H_\Lambda(J, \sigma_\Lambda, \sigma_{\Lambda^c})$ obtained by changing the spin configuration and not only one random variable. This can be done and gives useful result in the case $\alpha = 3 + \epsilon$. In the $\alpha = 2 + \epsilon$, $\epsilon < 1$ or in the three-dimensional case the arguments used in this paper do not give useful results. A new method has to be found.

2. DESCRIPTION OF THE METHOD AND MAIN RESULTS

One considers the classical x–y spin glass model in two dimensions, the Hamiltonian of which is given by the following: If Λ is a finite box

$$H_\Lambda(\theta_\Lambda, \theta_{\Lambda^c}) = -\sum_{x \in \Lambda} \sum_{y \in \mathbb{Z}^2 \setminus \{x\}} \frac{J_{xy}}{|x-y|^{3+\epsilon}} \cos(\theta_x - \theta_y) \tag{2.1}$$

where $\theta_\Lambda = (\theta_x)_{x \in \Lambda}$ is a configuration. For any x, θ_x belongs to the torus Π. We assume that $(J_{xy})_{(x,y) \in \mathbb{Z}^4}$ are independent identically distributed random variables with mean zero. We assume also that the J and the θ are independent. For the sake of simplicity we will assume that $J_{xy} = \pm 1$ with probability $1/2$.

The result of this paper is the following:

Theorem 2.1. Let $P(J)$ be any external Gibbs state corresponding to the Hamiltonian (2.1). Then for almost all J, $P(J)$ is invariant by rotation.

One remarks also that the Theorem 2.1 implies

$$\langle \cos\theta_x \rangle(J) = \int P(J)(d\theta)\cos\theta_x = 0$$

for almost all J and in particular

(i) $$\mathbb{E}(\langle \cos\theta_x \rangle_{(J)}) = 0$$

(ii) $$q_{\mathrm{EA}} = \mathbb{E}\big(\langle \cos\theta_x \rangle^2_{(J)}\big) = 0$$

The proof is based on the following arguments which come from Pfister[21] and Fröhlich and Pfister.[22]

Let P be an external Gibbs state and P_Λ be the Gibbs distribution in a finite volume Λ given a boundary condition θ_{Λ^c} that is

$$P_\Lambda(d\theta_\Lambda, \theta_{\Lambda^c}) = \frac{\exp - \beta H_\Lambda(\theta_\Lambda, \theta_{\Lambda^c}) \prod_{x\in\Lambda} d\theta_x}{Z_\Lambda(\theta_{\Lambda^c})} \tag{2.2}$$

Giving $a \in \Pi^{\mathbb{Z}^2}$, $a = (a_x)_{x\in\mathbb{Z}^2}$ in such a way that $a_x = 0$ except in a finite subset of $\mathbb{Z}^2$, say Λ_0 then one defines $\theta + a$ as $(\theta + a)_x = \theta_x + a_x$ $\forall x \in \mathbb{Z}^2$, and let $\tau_a P$ be the image of P by the map $\theta \to \theta + a$. It is clear that $\tau_a P$ is absolutely continuous with respect to P. Moreover

$$\frac{d\tau_a P}{dP} = \exp \beta \sum_{\substack{x, y \\ x\neq y}} \frac{J_{xy}}{|x-y|^{3+\epsilon}} \big[\cos(\theta_x - \theta_y - a_x + a_y) - \cos(\theta_x - \theta_y)\big] \tag{2.3}$$

If Λ_0 is finite the sum in the right-hand side of (2.3) is bounded since $a_x - a_y = 0$ if $\{x, y\}$ belongs to Λ_0^c. The relative entropy $\tau_a P$ with respect to P is given by

$$S(\tau_a P/P) = -\int P(d\theta)\log\left(\frac{d\tau_a P}{dP}\right) = \langle \tau_a H - H \rangle_P \tag{2.4}$$

One can look instead at the relative entropy of $\tau_a P \otimes \tau_{-a} P$ with respect to $P \otimes P$; this gives

$$S(\tau_a P \otimes \tau_{-a} P / P \otimes P) = \langle \tau_a H + \tau_{-a} H - 2H \rangle_P \tag{2.5}$$

One remarks

$$S(\tau_a P \otimes \tau_{-a} P / P \otimes P) = S(\tau_a P/P) + S(\tau_{-a} P/P) \tag{2.6}$$

By the Jensen inequality $S(\tau_a P/P) \geqslant 0$ and $S(\tau_{-a} P/P) \geqslant 0$; therefore if $S(\tau_a P/P) + S(\tau_{-a} P/P) \leqslant k$ we get $S(\tau_a P/P) \leqslant k$.

Now we choose the $(a_x)_{x\in\mathbb{Z}^2}$ in the following way: Let $|x| = \mathrm{Max}(|x_1|, |x_2|)$ if $x = (x_1, x_2) \in \mathbb{Z}^2$. We rotate *all* the spins θ_x which belong to a square Λ_l centered at the origin (defined by $|x| \leqslant l$) by an arbitrary $t \in \Pi$.

On each crown (defined by $|x| = \text{const.}$) we rotate the spins θ by

$$a_x = a_{|x|} = \frac{t}{F(l,L)} \sum_{k=|x|}^{L} \frac{1}{k} \qquad \text{if} \quad l < |x| \leqslant L \tag{2.7}$$

where

$$F(l,L) = \sum_{k=l}^{L} 1/k$$

$$a_x = 0 \qquad \text{if} \quad |x| > L$$

Call $a_{l,L}$ such a rotation.

If we can prove that almost surely (with respect to J) and uniformly with respect to l

$$\lim_{L\to\infty} S(\tau_{al,L} P/P) = 0 \tag{2.8}$$

then $\lim_{L\to\infty} \tau_{al,L} P$ is absolutely continuous with respect to P (by the Jensen inequality and the fact that (2.8) implies: there exists a measurable function $k(J)$ such that $\text{Prob}[k(J) = \infty] = 0$ and $S(\tau_{al,L} P/P) \leqslant k(J)$.

On the other hand $\lim_{L\to\infty} \tau_{al,L} P$ restricted to Λ_l coincides with the Gibbs states $\hat{P}$ obtained by turning *all* the spins of an angle t, therefore $\hat{P}$ is absolutely continuous with respect to P. Since P is extremal, this implies $P = \hat{P}$. This is Theorem 1.

Instead of proving (2.8), we prove the following proposition which implies (2.8):

Proposition 2.2. Uniformly with respect to l, *uniformly with respect to θ*, almost *surely* with respect to J

$$\lim_{L\to\infty} \Delta H(a_{l,L}) = \lim_{L\to\infty} \frac{1}{4} \left[-\tau_{al,L} H - \tau_{-al,L} H + 2H \right] = 0 \tag{2.9}$$

Remark. The crucial fact is the uniformity with respect to θ. Nonuniform results are trivial but useless.

One remarks that $\Delta H(a_{l,L})$ can be written as

$$\Delta H(a_{l,L}) = -\sum_{\substack{x,y \\ x\neq y}} \frac{J_{xy}}{|x-y|^{3+\epsilon}} \cos(\theta_x - \theta_y) \sin^2\left(\frac{a_n - a_y}{2}\right) \tag{2.10}$$

because

$$\cos(\theta + a) + \cos(\theta - a) - 2\cos(\theta) = 4\cos\theta \sin^2\left(\frac{a}{2}\right)$$

Since

$$a_x = 0 \qquad \text{if} \quad x \in \Lambda_L^c = \{x \in \mathbb{Z}^2/|x| > L\}$$
$$a_x = a_y \qquad \text{if} \quad x \in \Lambda_l$$

we get

$$\Delta H(a_{l,L}) = 2 \sum_{\substack{x \in \Lambda_l \\ y \in \Lambda_l^c}} + \sum_{\substack{x \in \Lambda_L \setminus \Lambda_l \\ y \in \Lambda_L \setminus \Lambda_l}} + 2 \sum_{\substack{x \in \Lambda_L \setminus \Lambda_l \\ y \in \Lambda_L^c}} \tag{2.11}$$

Therefore, if Λ_1 and Λ_2 are two boxes, we will define

$$\Delta H(\theta_{\Lambda_1}, \theta_{\Lambda_2}) = -\sum_{x \in \Lambda_1} \sum_{y \in \Lambda_2 \setminus \{x\}} \frac{J_{xy}}{|x-y|^{3+\epsilon}} \cos(\theta_x - \theta_y) \sin^2\left(\frac{a_x - a_y}{2}\right) \tag{2.12}$$

The strategy of the proof of the Proposition 2.2 is the following:

Step 1

We prove that we can restrict in (2.11) the two sums

$$\sum_{\substack{x \in \Lambda_l \\ y \in \Lambda_l^c}} \quad \text{and} \quad \sum_{\substack{x \in \Lambda_L \setminus \Lambda_l \\ y \in \Lambda_L^c}}$$

to finite volume one; this comes from the following lemma:

Lemma 2.3. Uniformly with respect to $J, \Lambda_2, \theta_{\Lambda_2}$ if $|\Lambda_1| = L$ and $\mathrm{dist}(\Lambda_1, \Lambda_2) \geqslant L^2$ then

$$|\Delta H(\theta_{\Lambda_1}, \theta_{\Lambda_2})| \leqslant K_1/L^{2\epsilon} \tag{2.13}$$

for some constant K_1. Therefore, we can assume $y \in \Lambda_{L^2+l}$ in the two previous sums.

Step 2

We prove that we can restrict the two previous sums to a smaller volume by using probability estimates. We prove the following lemma:

Lemma 2.4. *Uniformly with respect to* $l, \theta(\Lambda_{L+l}), \theta(\Lambda_{L^2+l} \setminus \Lambda_{2L+l})$ and almost surely with respect to J

$$\lim_{L \to \infty} |\Delta H(\theta(\Lambda_{L+l}), \theta(\Lambda_{L^2+l} \setminus \Lambda_{2L+l}))| = 0 \tag{2.14}$$

Step 3

We are reduced to estimate the contribution of

$$\Delta H\big(\theta(\Lambda_{L+l}),\theta(\Lambda_{2L+l})\big)$$

To do this, one decomposes crowns $\Lambda_{2L+l}\backslash\Lambda_{L+l}$ and $\Lambda_{L+l}\backslash\Lambda_l$ into crown of width $L/2, L/2^2, \dots, L/2^K$, respectively, and K will be chosen such that $L/2^K = O(\log L)$. This construction will be recursive. At the end we get a family of crowns $\{\mathscr{C}_i^{(K)},\ i=1,\dots,2^K+1\}$ centered at the origin. We get the following hierarchy:

$$\begin{aligned}\Delta H\big(\theta(\Lambda_{L+l}),\theta(\Lambda_{2L+l})\big)\\ &= 2\Delta H\big(\theta(\Lambda_l),\theta(\mathscr{C}_1^{(K+1)})\big)\\ &\quad+\sum_{j=1}^{2^{K+1}}\left\{\Delta H\big(\theta(\mathscr{C}_j^{(K+1)}),\theta(\mathscr{C}_j^{(K+1)})\big)+2\Delta H\big(\theta(\mathscr{C}_j^{(K+1)})\theta(\mathscr{C}_{j+1}^{(K+1)})\big)\right\}\\ &\quad+\sum_{p=2}^{K}R_p\end{aligned}\tag{2.15}$$

for a given p, R_p corresponds to terms as

$$\Delta H\big(\theta(\mathscr{C}_j^{(p)}),\theta(\mathscr{C}_{j+1}^{(p)})\big)\quad\text{or}\quad\Delta H\big(\theta(\mathscr{C}_j^{(p)})\theta(\mathscr{C}_{j+1}^{(p)})\big)$$

which are smaller than those corresponding to adjacent crowns. We prove the following lemma:

Lemma 2.5. Uniformly with respect to l and $\theta(\Lambda_{2L+l})$ and almost surely with respect to J

$$\lim_{L\to\infty}\sum_{p=2}^{K}R_p=0\qquad\text{if}\quad K=O(\log L)\tag{2.16}$$

Step 4

The last step consists in estimating the first three terms of the right-hand side of (2.15). This corresponds to a "classical $(x\text{–}y)$ model with interaction only between adjacent crowns of width $\log L$." For this model, we prove by using an L^∞ estimate the following lemma.

Lemma 2.6. There exists a constant K_2 and for any l a constant $L_0(l)$ such that uniformly with respect to θ and J

$$|\Delta H|\leqslant K_2(\log L)^{-\epsilon}\qquad\text{if}\quad L\geqslant L_0(l)\tag{2.17}$$

Therefore

$$\lim_{L\to\infty}|\Delta H|=0\tag{2.18}$$

3. PROOFS OF THE PREVIOUS LEMMAS

Step 1

Proof of Lemma 2.3. This is simply

$$|\Delta H(\theta(\Lambda_1), \theta(\Lambda_2))| \leqslant \sum_{x \in \Lambda_1} \sum_{y\,:\,|y| \geqslant L^2} \frac{1}{|y|^{3+\epsilon}} \leqslant L^2 \times \frac{K_1}{(L^2)^{1+\epsilon}} = K_1 L^{-2\epsilon} \tag{3.1}$$

for some constant K_1.

Step 2

Step 2 is based on the following classical probability estimates of large deviation of sum of independent sub-Gaussian variables.

Lemma 3.1 (Bernstein inequality). Let $X_1, \ldots, X_n$ be n sub-Gaussian independent random variables; then

$$\text{Prob}\left\{ \left| \sum_{i=1}^{n} X_i \right| \geqslant t \left[\sum_{i=1}^{n} E(X_i^2) \right]^{1/2} \right\} \leqslant 2e^{-t^2/2} \tag{3.2}$$

Now, the trick to be used is the discretization of the θ.

We expand θ_x into dyadic expansion. From the measure theoretical point of view this is an isomorphism:

$$\theta_x = \sum_{n=1}^{\infty} \frac{\lambda_{x,n}}{2^n} \qquad \text{with} \quad \lambda_{x,n} \in \{0, 1\}$$

and the Lebesgue measure on $\Pi \sim [0, 1]$ is nothing but the product measure on $\{0, 1\}^{\mathbb{N}^*}$ given by $p(0) = p(1) = 1/2$. For a given M one defines

$$\theta_x^{(M)} = \sum_{n=1}^{M} \frac{\tau_{x,n}}{2^n} \qquad \text{clearly} \quad |\theta_x - \theta_x^{(M)}| \leqslant \frac{1}{2^M} \tag{3.3}$$

Therefore

$$\begin{aligned} |\cos(\theta_x - \theta_y) - \cos(\theta_x^{(M)} - \theta_y^{(M)})| &\leqslant 2\left|\sin\left(\frac{\theta_x - \theta_x^{(M)} - \theta_y + \theta_y^{(M)}}{2}\right)\right| \\ &\leqslant 2^{1-M} \end{aligned} \tag{3.4}$$

Thus, if

$$\Delta H\big(\theta^{(M)}(\Lambda_1),\theta^{(M)}(\Lambda_2)\big) = -\sum_{\substack{x\in\Lambda_1\\ y\in\Lambda_2}} \frac{J_{xy}}{|x-y|^{3+\epsilon}}\cos\big(\theta_x^{(M)}-\theta_y^{(M)}\big)\sin^2\Big(\frac{a_x-a_y}{2}\Big) \tag{3.5}$$

one gets

$$|\Delta M(\theta(\Lambda_1),\theta(\Lambda_2)) - \Delta M\big(\theta^{(M)}(\Lambda_1),\theta^{(M)}(\Lambda_2)\big)| \leqslant \text{const.}|\Lambda_1|2^{1-M} \tag{3.6}$$

In particular in the case where $\Lambda_1 = \Lambda_{(L+l)}$, we choose M such that

$$|\Lambda_{L+l}|2^{1-M} \leqslant (\log L)^{-\epsilon} \tag{3.7}$$

and we can assume $l \leqslant L$; a choice is

$$M = \left[\frac{\log\big(4L^2(\log L)^{\epsilon}}{\log 2}\right]$$

We remark that $M = O(\log L)$. Therefore, it is sufficient to prove Lemma 2.4 with discretized $\theta^{(M)}$ where $M = O(\log L)$. In order to avoid involved notation, we prove Lemma 2.4 with $\theta^{(M)}(\Lambda_{L^2+l}\backslash\Lambda_{2L+l})$ is replaced by $\theta^{(M)}(\Lambda_{(L+l)^2}\backslash\Lambda_{2(L+l)})$.

Let $L_1 = L + l$, we subdivide the box $\Lambda_{L_1^2}$ into boxes Λ_i of side $2L_1$. Λ_{L_1} will be the centered one. We get

$$\Delta H\big(\theta^{(M)}(\Lambda_{L_1}),\theta^{(M)}(\Lambda_{L_1^2}\backslash\Lambda_{2L_1}\big) = \sum_{\Lambda_i\subset\Lambda_{L_1^2}\backslash\Lambda_{2L_1}} \Delta H\big(\theta^{(M)}(\Lambda_{L_1}),\theta^{(M)}(\Lambda_i)\big) \tag{3.8}$$

Now we estimate $\Delta M(\theta^{(M)}(\Lambda_{L_1}),\theta^{(M)}(\Lambda_i))$.

Lemma 3.2. There exists a constant K_2 such that

$$\begin{aligned}&\text{Prob}\Big[\text{at least for one configuration pair } \theta^{(M)}(\Lambda_{L_1})\theta^{(M)}(\Lambda_i)\\ &\qquad |\Delta H\big(\theta^{(M)}(\Lambda_{L_1}),\theta^{(M)}(\Lambda_i)\big)| \geqslant L_1^{3+\delta}/|z_i|^{3+\epsilon}\Big]\\ &\leqslant 2(2^M)^{2L_1^2}\exp - L_1^{2+2\delta}/K_2^2\end{aligned} \tag{3.9}$$

where z_i is the center of Λ_i and δ is any real positive number.

Proof. Let us fix the configuration $\theta^{(M)}(\Lambda_{L_1}),\theta^{(M)}(\Lambda_i)$ and define the random variables: if $x \in \Lambda_{L_1}$ and $y \in \Lambda_i$

$$\eta(x,y) = \frac{J_{xy}}{|x-y|^{3+\epsilon}}\cos\big(\theta_x^{(M)}-\theta_y^{(M)}\big)\sin^2\Big(\frac{a_x-a_y}{2}\Big)|z_i|^{3+\epsilon} \tag{3.10}$$

Then

$$\Delta H\big(\theta^{(M)}(\Lambda_{L_1}),\theta^{(M)}(\Lambda_i)\big) = \frac{1}{|z_i|^{3+\epsilon}} \sum_{\substack{x\in\Lambda_{L_1}\\ y\in\Lambda_i}} \eta(x,y) \tag{3.11}$$

It is easy to see that $\mathbb{E}(\eta)=0$, $\mathbb{E}(\eta^2)\leqslant K_2^2$, and $|\eta|\leqslant K_3$ for some constants K_2, K_3. Therefore, the random variables η are sub-Gaussian random variables and the variance of $\sum_{x\in\Lambda_{L_1}, y\in\Lambda_i}\eta(x,y)$ satisfies the following estimate:

$$D = \sum_{\substack{x\in\Lambda_{L_1}\\ y\in\Lambda_1}} \mathbb{E}\big(\eta^2(x,y)\big) \leqslant K_2^2 L_1^4$$

Now if we choose t in the Lemma 3.1, as $t = L_1^{3+\delta}/\sqrt{D}$ we get $t \geqslant L_2^{1+\delta}/K_2$ and

$$\operatorname{Prob}\left\{\left|\sum_{\substack{x\in\Lambda_{L_1}\\ y\in\Lambda_i}} \eta(x,y)\right| \geqslant L_1^{3+\delta}\right\} \leqslant 2\exp - \frac{L_1^{2+2\delta}}{K_2^2} \tag{3.12}$$

Now, using the fact that the number of configuration pairs $\theta^{(M)}(\Lambda_{L_1})$, $\theta^{(M)}(\Lambda_i)$ is bounded above by $(2^M)^{2L_1^2}$ and the subadditivity of the probability measure we get that the left-hand side of (3.9) does not exceed

$$(2^M)^{2L_1^2}\exp - L_1^{2+2\delta}/K_2^2 \quad \blacksquare$$

Remark. It is at this step that the discretization of the angle is useful. The fact that M is $O(\log L)$ is *crucial* in order to obtain an arbitrarily small probability of the previous events in the limit $L_1 \to \infty$. The choice $1/|x-y|^{3+\epsilon}$ in (2.1) is done in order to obtain (1) a decreasing energy between two blocks (by choosing $\delta < \epsilon$) if the distance between blocks have the same order as the side of the block, (2) a probability estimate uniform with respect to θ. Similar estimates are obtained in the case $M=1$ by Khanin and Sinaï.[16]

In order to prove Lemma 2.4, we need the following *crucial* fact which would be used constantly in the sequel.

Lemma 3.3. Let $(\Lambda_{1i})_{i=1\ldots N_1}$ and $(\Lambda_{2j})_{j=1\ldots N_2}$ be two families of disjoint subsets of $\mathbb{Z}^2$. Call

$$\Lambda_1 = \bigcup_{i=1}^{N_1} \Lambda_{1i} \quad \text{and} \quad \Lambda_2 = \bigcup_{j=1}^{N_2} \Lambda_{2j}$$

If, for any i and any j

$$\text{Prob}\Big(\text{at least for one configuration pair } \theta^{(M)}(\Lambda_{1i}), \theta^{(M)}(\Lambda_{2j})$$
$$|\Delta H\big(\theta^{(M)}(\Lambda_{1i}), \theta^{(M)}(\Lambda_{2j})| \geqslant \alpha_{ij}\Big) \leqslant \epsilon \tag{3.13}$$

Then

$$\text{Prob}\Bigg[\text{at least for one configuration pair } \theta^{(M)}(\Lambda_1), \theta^{(M)}(\Lambda_2)$$
$$|\Delta H\big(\theta^{(M)}(\Lambda_1), \theta^{(M)}(\Lambda_2)\big)| \geqslant \sum_{i,j} \alpha_{ij}\Bigg] \leqslant N_1 N_2 \epsilon \tag{3.14}$$

Proof. The events in (3.14) are

$$A = \bigcup_{\theta^{(M)}(\Lambda_1)\theta^{(M)}(\Lambda_2)} \left\{ J \Big/ \left| \sum_{i,j} \Delta H\big(\theta^{(M)}(\Lambda_{1i}), \theta^{(M)}(\Lambda_{2j})\big) \right| \geqslant \sum_{i,j} \alpha_{ij} \right\} \tag{3.15}$$

If we can prove that A is contained in B, where

$$B = \bigcup_{i,j} \bigcup_{\theta^{(M)}(\Lambda_{1i}), \theta^{(M)}(\Lambda_{2j})} \left\{ J/|\Delta H\big(\theta^{(M)}(\Lambda_{1i}), \theta^{(M)}(\Lambda_{2j})| \geqslant \alpha_{ij} \right\} \tag{3.16}$$

The subadditivity of the probability measure implies (3.14).

We prove B^c is contained in A^c. Let J be an element of B^c; then, for any $i, j, \theta^{(M)}(\Lambda_{1i}), \theta^{(M)}(\Lambda_{2j})$ we get

$$|\Delta H\big(\theta^{(M)}(\Lambda_{1i}), \theta^{(M)}(\Lambda_{2j})\big)(J)| \leqslant \alpha_{ij} \tag{3.17}$$

Therefore, for any configuration pair $\theta^{(M)}(\Lambda_1), \theta^{(M)}(\Lambda_2)$, the following inequality is true:

$$|\Delta H\big(\theta^{(M)}(\Lambda_1), \theta^{(M)}(\Lambda_2)\big)| \leqslant \sum_{i,j} \alpha_{ij} \tag{3.18}$$

and J belongs to A^c. The lemma is proved. ■

Proof of Lemma 2.4. Using Lemmas 3.2 and 3.3, we get

$$\text{Prob}\Bigg[\text{at least for one configuration pair } \theta^{(M)}(\Lambda_{L_1}), \theta^{(M)}(\Lambda_{L_1^2} \backslash \Lambda_{2L_1})$$
$$|\Delta H\big(\theta^{(M)}(\Lambda_{L_1}), \theta^{(M)}(\Lambda_{L_1^2} \backslash \Lambda_{2L_1})\big)| \geqslant L_1^{3+\delta} \sum_i \frac{1}{|z_i|^{3+\epsilon}}\Bigg]$$
$$\leqslant 2(L_1)^2 (2^M)^{2L_1^2} \exp - L_1^{2+2\delta}/K_2^2 \tag{3.19}$$

Now, $|z_i|$ is equal to $L_1|p_i|$, where p_i are the centers of squares with side 2; therefore $\sum_i 1/|z_i|^{3+\epsilon} \leqslant K_4 L_1^{-3-\epsilon}$ for some constant K_4.

If we choose $0 < \delta < \epsilon$, we get that

$$\text{Prob}\Big[\text{at least for one configuration pair } \theta^{(M)}(\Lambda_{L_1}), \theta^{(M)}(\Lambda_{L_1^2}\backslash\Lambda_{2L_1})$$

$$|\Delta H(\theta^{(M)}(\Lambda_{L_1}), \theta^{(M)}(\Lambda_{L_1^2}\backslash\Lambda_{2L_1}))| \geqslant K_4 L_1^{\delta-\epsilon}\Big]$$

is smaller than $K_5 \exp - L_1^{2+2\delta}/2K_2^2$ for some constant K_5 if L_1 is big enough. Since $\sum_{L_1=1}^{\infty} \exp - L_1^{2+2\delta}/2K_2^2 < \infty$, the first Borel Cantelli lemma implies Lemma 2.4. ■

Step 3

Step 3 is based on the recursive subdivision of crowns shown in Fig. 1. One defines

$$\mathscr{C}_1^{(0)} = \Lambda_{2L+l}\backslash\Lambda_l, \qquad \mathscr{C}_1^{(1)} = \Lambda_{L+l}\backslash\Lambda_l, \qquad \mathscr{C}_2^{(1)} = \Lambda_{2L+l}\backslash\Lambda_{L+l}$$

Now one defines recursively

$$\mathscr{C}_i^{(K)}, \qquad i = 1, \ldots, 2^K$$

by

$$\mathscr{C}_i^{(K)} = \mathscr{C}_{2i-1}^{(K+1)} \cup \mathscr{C}_{2i}^{(K+1)}$$

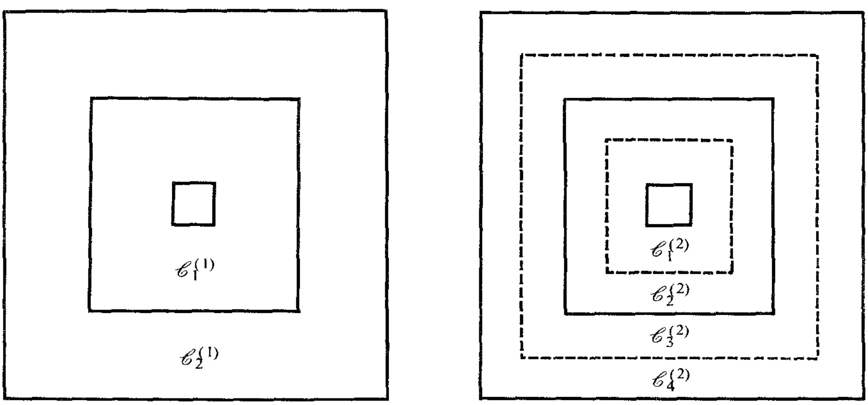

Fig. 1.

We obtain the following hierarchy:

$$\begin{aligned}\Delta M\big((\Lambda_{L+l}),\theta(\Lambda_{2L+l})\big)&=2\Delta H\big(\theta(\Lambda_l),\theta(\mathscr{C}_1^{(K+1)})\big)\\&\quad+\sum_{j=1}^{2^K}\big\{\Delta H\big(\theta(\mathscr{C}_j^{(K+1)}),\theta(\mathscr{C}_j^{(K+1)})\big)+2\Delta H\big(\theta(\mathscr{C}_j^{(K+1)}),\theta(\mathscr{C}_{j+1}^{(K+1)})\big)\big\}\\&\quad+\sum_{p=2}^{K}R_p\end{aligned}\tag{3.20}$$

where

$$\begin{aligned}R_p&=\sum_{J=1}^{2^{p-1}}2\Delta H\Big(\theta\big(\mathscr{C}_j^{(p)}\big),\theta\big(\mathscr{C}_{j+2}^{(p)}\big)\Big)+\sum_{j=1}^{2^{p-2}}2\Delta H\Big(\theta\big(\mathscr{C}_{2j-1}^{(p)}\big),\theta\big(\mathscr{C}_{2j+2}\big)\Big)\\&\quad+\sum_{J=2}^{2^{p-1}+2}2\Delta H\big(\theta(\Lambda_l),\theta(\mathscr{C}_j^{(p)})\big)\end{aligned}\tag{3.21}$$

In order to apply probability estimates, we first discretize the θ as in Step 2, but with different M: Let $\tilde{R}_p$ be the two first sums in (3.21) and $\tilde{R}_p^{(M)}$ the same sums but with discretized θ. We get the following lemma:

Lemma 3.4. If we choose $K=[\log[L(\log L)^{-1}]/\log 2]$ and if $l\leqslant(\log L)^{1/2}$ then, for some constants K_6 and K_2

$$\sum_{p=1}^{K}|\tilde{R}_p-\tilde{R}_p^{(M)}|\leqslant K_6\frac{L^{1-\epsilon}}{2^{M-1}(\log L)^2}\tag{3.22}$$

$$\sum_{p=2}^{K}\left\|\Delta H\left(\theta(\Lambda_l),\bigcup_{j=2}^{2^{p-1}+2}\theta\big(\mathscr{C}_j^{(p)}\big)\right)\right\|\leqslant K_7(\log L)^{-\epsilon}\tag{3.23}$$

Proof. Let B_n be the crown defined by $|x|=n$. It is straightforward that

$$\begin{aligned}&|\Delta H\Big(\theta(\mathscr{C}_j^{(p)}),\theta\big(\mathscr{C}_{j+2}^{(p)}\big)\Big)-\Delta H\Big(\theta^{(M)}(\mathscr{C}_j^{(p)}),\theta^{(M)}\big(\mathscr{C}_{j+2}^{(p)}\big)\Big)\\&\quad\leqslant\frac{1}{2^{M-1}\big[F(l,L)\big]^2}\sum_{q=jL2^{-p}+l}^{(j+1)L2^{-p}+l}\sum_{K=1}^{L/2^p}\left\{\sum_{\substack{x\in B_q\\y\in B_K+l+(j+2)L2^{-p}}}\frac{1}{|x-y|^{3+\epsilon}}\right\}\\&\quad\times\left\{\sum_{s=q}^{K+(j+2)L2^{-p}+l}1/s\right\}^2\end{aligned}\tag{3.24}$$

Now

$$\sum_{\substack{x \in B_q \\ y \in B_K + (j+2)L2^{-p} + l}} \frac{1}{|x-y|^{3+\epsilon}} \leqslant K_8 q\left[K + (j+2)L2^{-p} + l - q\right]^{-2-\epsilon} \tag{3.25}$$

and

$$\left(\sum_{s=q}^{K+(j+2)L2^{-p}+l} 1/s\right)^2 \leqslant q^{-2}\left[K + (j+2)L/2^p + l - q\right]^2 \tag{3.26}$$

If we remark that in the right-hand side of (3.24) q is less than $l + (j+1)L2^{-p}$ the last three sums in the left-hand side of (3.24) are less than

$$\sum_{K=1}^{L/2^p} \left[K + (j+2)L2^{-p} + l - q\right]^{-\epsilon} \leqslant K_9 (L2^{-p})^{1-\epsilon} \tag{3.27}$$

Collecting all these estimates, the left-hand side of (3.24) is less than

$$K_{10} \frac{j^{-1}(L2^{-p})^{1-\epsilon}}{2^{M-1}\left[F(l,L)\right]^2} \tag{3.28}$$

Therefore

$$|\tilde{R}_p - \tilde{R}_p^{(M)}| \leqslant \frac{K_{11} p (L2^{-p})^{1-\epsilon}}{2^{M-1}\left[F(l,L)\right]^2} \tag{3.29}$$

Summing on p leads to (3.22). We estimate now

$$\left|\Delta H\left(\theta(\Lambda_l), \bigcup_{j=2}^{2^{p-1}+2} \theta(\mathscr{C}_j^{(p)})\right)\right|$$

It is straightforward that

$$\left|\Delta H\left(\theta(\Lambda_l), \bigcup_{j=2}^{2^{p-1}+2} \theta(\mathscr{C}_j^{(p)})\right)\right| \leqslant K_{12} l^2 (L2^{-p})^{-1-\epsilon} \tag{3.30}$$

Therefore, the right-hand side of (3.23) does not exceed $K_{13} l^2 (L/2^K)^{-1-\epsilon}$ which together with our hypothesis implies (3.23). The lemma is proved. ∎

This lemma, with the choice $M = [(1-\epsilon)\log L/\log 2]$ implies that

$$\sum_{p=1}^{K} |\tilde{R}_p - \tilde{R}_p^M| \leqslant 2K_6 (\log L)^{-2}$$

and we have to prove Lemma 2.5. With discretized θ. We consider the first sum in (3.21); the proof for the second sum is done along the same lines.

We first subdivide each crown $\mathscr{C}_j^{(p)}, \mathscr{C}_{j+2}^{(p)}$ in squares $\mathscr{C}_{j,q}^{(p)}$ and $\mathscr{C}_{j+2,r}^{(p)}$ of width $2L2^{-p}$; each crown is not exactly subdivided, because there is the centered box Λ_l. On each crown, there are four rectangles of side $2l \times 2L2^{-p}$ we call them also $\mathscr{C}_{j,q}^{(p)}, \mathscr{C}_{j+2,2}^{(p)}$. As we will see later from a probabilistic point of view, the distinction between squares and rectangles is irrelevant. Notice that there are $4(2j-1)$ squares in each crown and four rectangles. See Fig. 2.

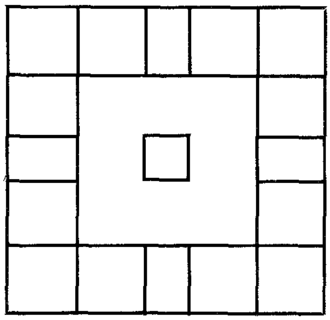

Fig. 2.

According to this decomposition $\Delta H(\theta^{(M)}(\mathscr{C}_j^p), \theta^{(M)}(\mathscr{C}_{j+2}^{(p)}))$ can be written as

$$\sum_{q=1}^{(8j)} \sum_{r=1}^{(8j+16)} \Delta H\left(\theta^{(M)}(\mathscr{C}_{j,q}^{(p)}), \theta^{(M)}(\mathscr{C}_{j+2,r}^{(p)})\right) \tag{3.31}$$

The following lemma is similar to Lemma 3.2, but here we have to take into account of the decrease which comes from the rotation a.

Lemma 3.5. If z_q and z_r denote the center of $\mathscr{C}_{j,q}^{(p)}, \mathscr{C}_{j+2,r}^{(p)}$, then, for some constants K_{14}, K_{15} and any $\delta > 0$

$$\text{Prob}\left\{ \begin{array}{l} \text{at least for one configuration pair } \theta^{(M)}(\mathscr{C}_{j,q}^{(p)}), \theta^{(M)}(\mathscr{C}_{j+2,r}^{(p)}) \\[2ex] |\Delta H(\theta^{(M)}(\mathscr{C}_{j,q}^{(p)}), \theta^{(M)}(\mathscr{C}_{j+2,r}^{(p)}))| \geqslant K_{15} \dfrac{(j)^{-2}(L2^{-p})^{3+\delta}(\log L)^{1/2}}{[F(l,L)]^2 |z_q - z_r|^{3+\epsilon}} \end{array} \right\}$$

$$\leqslant 2(2^M)^{2(L2^{-p})^2} \exp\left[-\frac{(L2^{-p})^{2+2\delta}}{K_{14}^2} \log L \right] \tag{3.32}$$

Proof of Lemma 3.5. We consider first the case where $\mathscr{C}_{j,q}^{(p)}, \mathscr{C}_{j+2,r}^{(p)}$ are squares. Let us fix the configuration $\theta^{(M)}(\mathscr{C}_{j,q}^{(p)}), \theta^{(M)}(\mathscr{C}_{j+2,r}^{(p)})$ and define the random variable: if $x \in \mathscr{C}_{j,q}^{(p)}$ and $y \in \mathscr{C}_{j+2,r}^{(p)}$

$$\eta(x, y) = \frac{J_{xy} \sin^2(a_x - a_y)\big[F(l, L)\big]^2}{|x-y|^{3+\epsilon}\big[\log(1+1/j)\big]^2} |z_q - z_r|^{3+\epsilon} \cos\big(\theta_x^{(M)} - \theta_y^{(M)}\big) \quad (3.33)$$

It is straightforward that

$$\mathbb{E}(\eta) = 0$$

$$\mathbb{E}(\eta^2) \leqslant K_{16} \operatorname*{Max}_{\substack{x \in \mathscr{C}_{j,q}^{(p)} \\ y \in \mathscr{C}_{j+2,r}^{(p)}}} \left| \sum_{K=|x|}^{|y|} 1/K \right|^4 \frac{1}{\big[\log(1+1/j)\big]^2} \leqslant K_{14}^2 \quad (3.34)$$

for some constant K_{14}. Therefore, the random variables η are sub-Gaussian random variables and the variance of

$$\sum_{\substack{x \in \mathscr{C}_{j,q}^{(p)} \\ y \in \mathscr{C}_{j+2,r}^{(p)}}} \eta(x, y)$$

is bounded above by $K_{14}^2(L2^{-p})^4$. Now if we choose t in Lemma 3.1 as $t = ((L2^{-p})^{3+\delta}/\sqrt{D})(\log L)^{1/2}$, we get

$$t \geqslant \frac{1}{K_{12}} (L2^{-p})^{1+\delta} (\log L)^{1/2}$$

and this lemma implies that

$$\operatorname{Prob}\left[\left|\sum_{\substack{x \in \mathscr{C}_{j,q}^{(p)} \\ y \in \mathscr{C}_{j+2,r}^{(p)}}} \eta(x, y)\right| \geqslant (L2^{-p})^{3+\delta} (\log L)^{1/2}\right]$$

$$\leqslant 2 \exp - \frac{(L2^{-p})^{2+2\delta} \log L}{K_{12}^2} \quad (3.35)$$

Now, using the fact that the number of configuration pairs $\theta^{(M)}(\mathscr{C}_{j,q}^{(p)})$ $\theta^{(M)}(\mathscr{C}_{j+2,r}^{(p)})$ is bounded above by $(2^M)^{2(L2^{-p})^2}$ and the subadditivity of the probability measure we get that the left-hand side of (3.32) does not exceed

$$2(2^M)^{2(L2^{-p})^2} \exp - \frac{(L2^{-p})^{2+2\delta} \log L}{K_{12}^2}$$

The lemma is proved if $\mathscr{C}_{j,q}^{(p)}$ and $\mathscr{C}_{j+2,r}^{(p)}$ are squares. If $\mathscr{C}_{j,q}^{(p)}$ or $\mathscr{C}_{j+2,r}^{(p)}$ are rectangles, the upper bound for the variance of $\sum \eta(x, y)$ is always true, the lower bound on t:

$$t \geqslant \frac{1}{K_{12}} (L2^{-p})^{1+\delta} (\log L)^{1/2}$$

is true and also (3.35). Therefore, the distinction between squares and rectangles is irrelevant. And the lemma is proved in all the cases. ■

Proof of Lemma 2.5. The Lemmas 3.3 and 3.5 lead to the fact that

$$\text{Prob}\Bigg[\text{at least for one configuration pair } \theta^{(M)}(\mathscr{C}_j^{(p)})\theta^{(M)}(\mathscr{C}_{j+2}^{(p)})$$

$$|\Delta H\big(\theta^{(M)}(\mathscr{C}_j^{(p)}), \theta^{(M)}(\mathscr{C}_{j+2}^{(p)})\big)|$$

$$\geqslant \frac{K_{14}(\log L)^{1/2}}{[F(l,L)]^2} (j)^{-2} (L2^{-p})^{3+\delta} \sum_{z_q, z_r} \frac{1}{|z_q - z_r|^{3+\epsilon}}\Bigg]$$

does not exceed

$$2(8j)(8j+16)\exp\left\{ M2(L2^{-p})^2 \log 2 - (\log L) \frac{(L2^{-p})^{2+2\delta}}{K_{12}^2} \right\} \tag{3.36}$$

It is straightforward that

$$\sum_{z_q, z_r} \frac{1}{|z_q - z_r|^{3+\epsilon}} \leqslant K_{15} \frac{j}{(L/2^p)^{3+\epsilon}} \tag{3.37}$$

and therefore

$$\sum_{j=1}^{2^{p-1}+1} \frac{(j)^{-2}}{[F(l,L)]^2} (L2^{-p})^{3+\delta} (\log L)^{1/2} \sum_{z_q, z_r} \frac{1}{|z_q - z_r|^{3+\epsilon}} \tag{3.38}$$

does not exceed

$$\frac{(\log 2^{p-1})(L2^{-p})^{\delta-\epsilon}(\log L)^{1/2}}{[F(l,L)]^2} \tag{3.39}$$

On the other hand

$$\sum_{j=1}^{2^{p-1}+1} (j)(j+2) \leqslant K_{16} 2^{3p}$$

for some constant K_{16}. Collecting these estimates and using Lemma 3.3, we obtain that

$$\text{Prob}\Bigg[\text{at least for one configuration } \theta^{(M)}\Bigg(\bigcup_{j=1}^{2^{p-1}+2} \mathscr{C}_j^{(p)}\Bigg)$$

$$\left|\sum_{j=1}^{2^{p-1}+2} 2\Delta H\big(\theta^{(M)}(\mathscr{C}_j^{(p)}),\theta^{(M)}(\mathscr{C}_{j+2}^{(p)})\big)\right|$$

$$\geqslant \frac{K_{15}(\log L)^{1/2}}{\left[F(l,L)\right]^2}\log 2^{p-1}\left(L2^{-p}\right)^{\delta-\epsilon}\Bigg]$$

does not exceed

$$K_{16}2^{3p}\exp\left[2M(L2^{-p})^2\log 2-\frac{(L2^{-p})^{2+2\delta}}{K_{12}^2}\log L\right] \tag{3.40}$$

Now

$$\sum_{p=2}^{K}(\log L)^{1/2}\frac{(\log 2^{p-1})}{\left[F(l,L)\right]^2}\left(L2^{-p}\right)^{\delta-\epsilon}$$

is bounded above by

$$K_{17}\frac{(K+2)(2^{-K}L)^{\delta-\epsilon}(\log L)^{1/2}}{\left[F(l,L)\right]^2} \tag{3.41}$$

Therefore, if we choose

$$K=\left[\frac{\log\left(L(\log L)^{-1}\right)}{\log 2}\right]$$

as in Lemma 3.4, the right-hand side of (3.42) does not exceed $K_{18}(\log L)^{-1/2+\delta-\epsilon}$. The subadditivity of the probability measure implies that

$$\text{Prob}\Bigg[\text{at least for one configuration } \theta^{(M)}\Bigg(\bigcup_{p=2}^{K}\bigcup_{j=1}^{2^{p-1}+2} \mathscr{C}_j^{(p)}\Bigg)$$

$$\left|\sum_{p=2}^{K}\sum_{j=1}^{2^{p-1}+2} 2\Delta H\big(\theta^{(M)}(\mathscr{C}_j^{(p)}),\theta^{(M)}(\mathscr{C}_{j+2}^{(p)})\big)\right| \geqslant K_{18}(\log L)^{-1/2+\delta-\epsilon}\Bigg]$$

does not exceed

$$K_{16}\sum_{p=2}^{K}2^{3p}\exp\left[2M(L2^{-p})^2\log 2-\frac{(L/2^{p})^{2+2\delta}}{K_{12}^2}(\log L)\right] \tag{3.42}$$

Remember that

$$K = \left[\frac{\log\left(L(\log L)^{-1}\right)}{\log 2} \right] \quad \text{and} \quad M = \left[(1-\epsilon) \frac{\log L}{\log 2} \right]$$

therefore $(L2^{-p}) \geqslant (L2^{-K}) \geqslant (\log L)$ and if L is big enough $(\log L)^{2\delta}/K_{12}^2 - 2(1-\epsilon)$ is bigger than $(\log L)^{2\delta}/2K_{12}^2$; therefore, the sum (3.42) is bounded above by

$$\sum_{p=1}^{K} 2^{3p} \exp - \frac{(L2^{-p})^2 (\log L)^{2+2\delta}}{2K_{12}^2}$$

which does not exceed

$$K 2^{3K} \exp - (L2^{-K})^2 \frac{(\log L)^{2+2\delta}}{2K_{12}^2} \leqslant (\log L)\left(\frac{L}{\log L} \right)^3 \exp - \frac{(\log L)^{2+2\delta}}{2K_{12}^2} \tag{3.43}$$

Now, if L is big enough, the right-hand side of (3.43) is bounded above by $\exp - (\log L)^{2+2\delta}/4K_{12}^2$, which is the general term of a summable series. Therefore, we can apply the Borel–Cantelli lemma and the lemma is proved. ■

Step 4

We are now at Step 4. We have to perform an L^∞ estimate of

$$2\Delta H\left(\theta(\Lambda_l), \theta(\mathscr{C}_1^{(K+1)})\right) + \sum_{j=1}^{2K} \left[\Delta H\left(\theta(\mathscr{C}_j^{(K+1)}), \theta(\mathscr{C}_j^{(K+1)})\right) + \Delta M\left(\theta(\mathscr{C}_j^{(K+1)}), \theta(\mathscr{C}_{j+1}^{(K+1)})\right) \right] \tag{3.44}$$

The first term is bounded above by

$$K_{17} \sum_{i=1}^{l} \sum_{j=l+1}^{[\log L]} \left[\sum_{\substack{x \in B_i \\ y \in B_j}} \frac{1}{|x-y|^{3+\epsilon}} \right] \frac{t^2}{[F(l,L)]^2} \left(\sum_{r=l}^{j} 1/r \right)^2 \tag{3.45}$$

It is straightforward that

$$\sum_{\substack{x \in B_i \\ y \in B_j}} \frac{1}{|x-y|^{3+\epsilon}} \leqslant K_8 i [j-i]^{-2-\epsilon} \tag{3.46}$$

and

$$\sum_{r=l}^{j} \frac{1}{r} \leqslant \frac{(j-l)}{l}$$

therefore, (3.45) is bounded above by

$$\frac{K_{17}t^2}{\left[F(l,L)\right]^2}\sum_{i=1}^{l}\frac{i}{l^2}\sum_{j=l+2}^{[\log L]}\frac{(j-l)^2}{(j-i)^{2+\epsilon}} \tag{3.47}$$

which is less than

$$\frac{K_{17}t^2}{\left[F(l,L)\right]^2}\sum_{j=l+1}^{[\log L]}\frac{1}{(j-l)^{\epsilon}} \leqslant \frac{K_{17}t^2}{\left[F(l,L)\right]^2}\left\{\left[(\log L)+l\right]\right\}^{1-\epsilon} \tag{3.48}$$

if $l \leqslant (\log L)^{1/2}$ as in Lemma 3.4 then (3.48) is bounded by

$$K_{18}(\log L)^{-1-\epsilon}$$

The sum in (3.44) is bounded above by

$$\frac{K_{19}t^2}{\left[F(l,L)\right]^2}\sum_{j=1}^{2^{K+1}}\left[\sum_{m=(j-1)L2^{-K-1}+l}^{jL2^{-K-1}+l}\;\sum_{n=m+1}^{(j+1)L2^{-K-1}+l}\right.$$

$$\left.\times\left\{\sum_{\substack{x\in B_m\\ y\in B_n}}\frac{1}{|x-y|^{3+\epsilon}}\right\}\left(\sum_{p=m}^{n}\frac{1}{p}\right)^2\right] \tag{3.49}$$

which, by (3.46), does not exceed

$$\frac{K_{19}t^2}{\left[F(l,L)\right]^2}\sum_{j=1}^{2^{K}}\left[\sum_{m=(j-1)L2^{-K-1}+l}^{jL2^{-K-1}+l}\frac{1}{m}\sum_{n=m+1}^{(j+1)L2^{-K-1}+l}\frac{1}{(n-m)^{\epsilon}}\right] \tag{3.50}$$

if we take into account the fact that $m \geqslant (j-1)L2^{-K-1}+l$

$$\sum_{n=m+1}^{(j+1)L2^{-K-1}+l}\frac{1}{(n-m)^{\epsilon}} = \sum_{n=1}^{(j+1)L2^{-K-1}+l-m}\frac{1}{n^{\epsilon}} \leqslant \sum_{n=1}^{L2^{-K}}\frac{1}{n^{\epsilon}}$$

which does not exceed $K_{20}(L2^{-K})^{1-\epsilon}$. Now

$$\sum_{j=1}^{2^{K+1}}\;\sum_{m=(j-1)L2^{-K-1}+l}^{jL2^{-K-1}+l}\frac{1}{m} = \sum_{j=l}^{L+l}\frac{1}{m}$$

which is less than $K_{21}\log((L+l)/l)$. Collecting all these estimates, we get

$$(3.47) \leqslant \frac{K_{22}t^2}{\left[F(l,L)\right]^2}\log\left(\frac{L}{l}+1\right)\left(L2^{-K}\right)^{1-\epsilon} \tag{3.51}$$

As in Lemma 3.4, $(L2^{-K}) = 0(\log L)$, therefore, $(3.49) \leqslant K_{22}t^2(\log L)^{-\epsilon}$. This proves the Lemma 2.6.

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